Cutoff frequency of experimentally generated noise: A Melnikov approach

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The motion of an overdamped particle excited by colored noise in a bistable potential is discussed in chaotic dynamics terms. A Melnikov approach is used to determine an appropriate cutoff frequency of experimentally generated noise as a function of noise correlation time. [S1063-651X(96)07909-3]

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The problem of crossing a potential barrier under stochastic excitation has been studied for many decades. Most of the theoretical investigations were conducted for the limit of white Gaussian noise. Various approaches to the more realistic case of colored noise have been found to yield contradictory results, especially for intermediate values of correlation time [1]. In this paper we approach this classical stochastic problem within the framework of chaotic dynamics. This approach does not yield an explicit expression for mean passage time. It offers, however, useful qualitative predictions, which may be important for experimental study. We start with the classical Langevin equation

$$\dot{x}(t)=-U'(x)+\xi(t), \tag{1}$$

where $U(x)$ is a potential with local maximum at $x=0$ and one or two local minima. Colored noise $\xi(t)$ of intensity D has an exponentially decaying autocorrelation function

$$\langle \xi(t)\xi(s)\rangle=\alpha D e^{-\alpha|t-s|}, \tag{2}$$

with correlation time $\tau=1/\alpha$. Such noise may be obtained by passing the standard white Gaussian noise $w(t)$ through a linear filter

$$\dot{\xi}(t)=-\alpha\xi(t)+\sqrt{2D}\alpha w(t). \tag{3}$$

Following Rice [2], Gaussian white noise $w(t)$ may be expressed as

$$w(t)=\lim_{N\to+\infty} w_N(t), \tag{4}$$

$$w_N(t)=N^{-1/2}\sum_{k=1}^{N}\sin(\omega_k t+\varphi_k), \tag{5}$$

where $\omega_k=k\omega_{\rm cut}/N$, $\omega_{\rm cut}$ is the cutoff frequency beyond which the power spectrum vanishes, and the random phases φ_k are uniformly distributed on the interval $[0,2\pi]$. In nearly all theoretical studies it is convenient—and common practice—to assume that there is no frequency cutoff (i.e., the cutoff frequency $\omega_{\rm cut}$ is infinity). In numerical simulations of Eq. (1) a cutoff frequency is implicit in the fact that the integration step is finite. In analog simulations of Eqs. (1) and (3) the correlation time τ is kept as small as possible to approximate closely the limit of white noise; however, $\omega_{\rm cut}$ is obviously finite. It has been commonly assumed in the literature, e.g., [3,4], that $\omega_{\rm cut}$ is sufficiently large if $\omega_{\rm cut}\gg\omega_0$, where $\omega_0=U''(x_{\rm min})$. The purpose of this paper is to show that in order to perform a reliable experiment $\omega_{\rm cut}$ should be compared with α, rather than with ω_0.

Using a substitution $y(t)=-U'(x)+\xi(t)$, Eqs. (1) and (3) may be rewritten in the form

$$\dot{x}(t)=y(t), \tag{6}$$

$$\dot{y}(t)=-[U''(x)+\alpha]y(t)-\alpha U'(x)+\sqrt{2D}\alpha w_N(t), \tag{7}$$

where in Eq. (7) we approximated $w(t)$ by $w_N(t)$ [see Eq. (5)] with N arbitrarily large but finite. Equations (6) and (7) describe a motion in the potential $\alpha U(x)$ of a particle with nonlinear damping $[U''(x)+\alpha]$ and excitation $\sqrt{2D}\alpha w_N(t)$. We note that this system may be investigated by using a Melnikov approach [5–8]. In what follows we assume $U(x)=-1/2x^2+1/4x^4$. Originally, the Melnikov approach was applied to systems with a perturbation consisting of two terms: a harmonic excitation $\lambda\sin\omega t$ and a linear damping $-\delta\dot{x}$, where λ and ω are the amplitude and the frequency, respectively, and $\delta>0$ is the linear damping coefficient [5]. The approach was applied to the case of nonlinear damping $\dot{x}(-\delta+\mu x^2)$ in [6] and to the case of quasiperiodic excitation $\lambda_1\sin(\omega_1 t+\phi_1)+\cdots+\lambda_N\sin(\omega_N t+\phi_N)$, where N is finite, in [7]. Based on the work of [7], the Melnikov approach was extended in [8] to multistable systems with stochastic excitation approximated by Eqs. (4) and (5). For each noise realization the expression for the Melnikov function corresponding to Eqs. (5), (6), and (7) is

$$F_M(t,t_1,t_2,\ldots,t_N)=-g(\alpha)+\sqrt{D}C(N,t,\alpha), \tag{8}$$

where, by applying the procedure introduced in [5–8], the function $g(\alpha)$ induced by the damping of Eq. (7) can be shown to be

$$g(\alpha)=4\alpha^{1/2}/3+28/15\alpha^{-1/2}, \tag{9}$$

and the function $C(N,t,\alpha)$ induced by the approximation $w_N(t)$ of the white noise is

$$C(N,t,\alpha)\equiv N^{-1/2}\sum_{k=1}^{N}S_M(\omega_k)\cos[\omega_k(t-t_k)+\varphi_k]. \tag{10}$$

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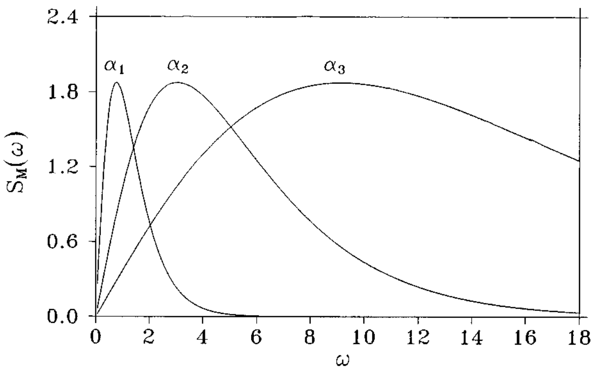

FIG. 1. Melnikov scale factor $S_M(\omega)$ for three different parameters α: $\alpha_1=1$, $\alpha_2=4^2$, and $\alpha_3=11^2$; see Eq. (11).

In Eq. (10)

$$S_M(\omega)=\sqrt{2}\pi\omega\ \mathrm{sech}(\pi\omega/2\sqrt{\alpha})/\sqrt{\alpha} \tag{11}$$

is referred to as the Melnikov scale factor [7]; see Fig. 1. The Melnikov necessary condition for the occurrence of chaotic jumps between the two wells is that F_M have simple zeros. This means that chaotic jumps over a barrier cannot be observed if F_M is negative for all t. It follows from the theory of chaotic transport [8] that the mean hopping rate increases as the average of the positive local maxima of the function F_M increases.

It is clear from Eqs. (8)–(10) that the contribution to F_M of white noise components with frequencies larger than ω_{cut} is negligible only if for those frequencies the scale factor $S_M(\omega)$ is negligibly small. The cutoff frequency should be selected accordingly. Failure to do so can significantly affect the mean hopping rate obtained in the experiment. The frequency for which $S_M(\omega)$ reaches its maximum is denoted by ω_M. Thus, noise components are most efficient in inducing jumps over a potential barrier if their frequency is close to the frequency ω_M. From Eq. (11) it follows that $\omega_M\approx\sqrt{\alpha}$; see Fig. 1. The cutoff frequency should therefore satisfy the inequality

$$\omega_{\mathrm{cut}}\geqslant c\sqrt{\alpha}, \tag{12}$$

where $c>1$ is sufficiently large. Then, all components with frequency larger than ω_{cut} have negligible contribution to the sum in Eq. (10); see Fig. 1 and Eq. (11). This condition may be stronger than the condition $\omega_{\mathrm{cut}}\gg\omega_0$ commonly used in experiments.

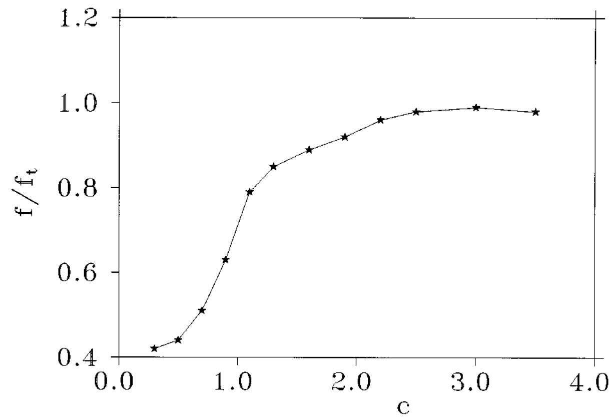

FIG. 2. The mean hopping rate f from numerical simulations of Eqs. (5), (6), and (7) vs the parameter c. Cutoff frequency $\omega_{\mathrm{cut}}=c\omega_M$ and f_t is the corresponding theoretical value.

For a given total power of noise pumped into a system and for a given noise correlation time $\tau=1/\alpha$, a cutoff frequency that does not satisfy Eq. (12) can yield a significantly different mean hopping rate than would be observed if Eq. (12) were satisfied. This was checked in numerical simulations of Eqs. (6) and (7) for the parameters $\alpha=1.6\times10^3$, $N=300$, $D=0.35$, and a few cutoff frequencies $\omega_{\mathrm{cut}}=c\omega_M$. In all cases the condition $\omega_{\mathrm{cut}}\gg\omega_0$ was satisfied; also, the noise spectrum was practically flat in the whole interval $[0,\omega_{\mathrm{cut}}]$. In Fig. 2 a mean hopping rate f normalized to the theoretical value f_t [9] is plotted as a function of parameter c. For each c the averaged rate corresponding to 100 different noise realizations was determined. From the dependence shown in Fig. 2 it is clear that for $c\geqslant2.5$ a cutoff frequency is acceptable; i.e., the experimental value of f is sufficiently close to its asymptotic value f_t. We note that if the cutoff frequency for an experiment is known, then the shortest possible noise correlation time τ consistent with Eq. (12) is $\tau\geqslant(c/\omega_{\mathrm{cut}})^2$.

Our conclusions are as follows: (1) the Melnikov approach makes it possible to select the cutoff frequency ω_{cut} so that errors associated with that selection are negligibly small; (2) depending upon the parameters of the problem, the criterion $\omega_{\mathrm{cut}}\gg\omega_0=U''(x_{\mathrm{min}})$ may be inadequate, especially if the noise correlation time τ is small.

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